

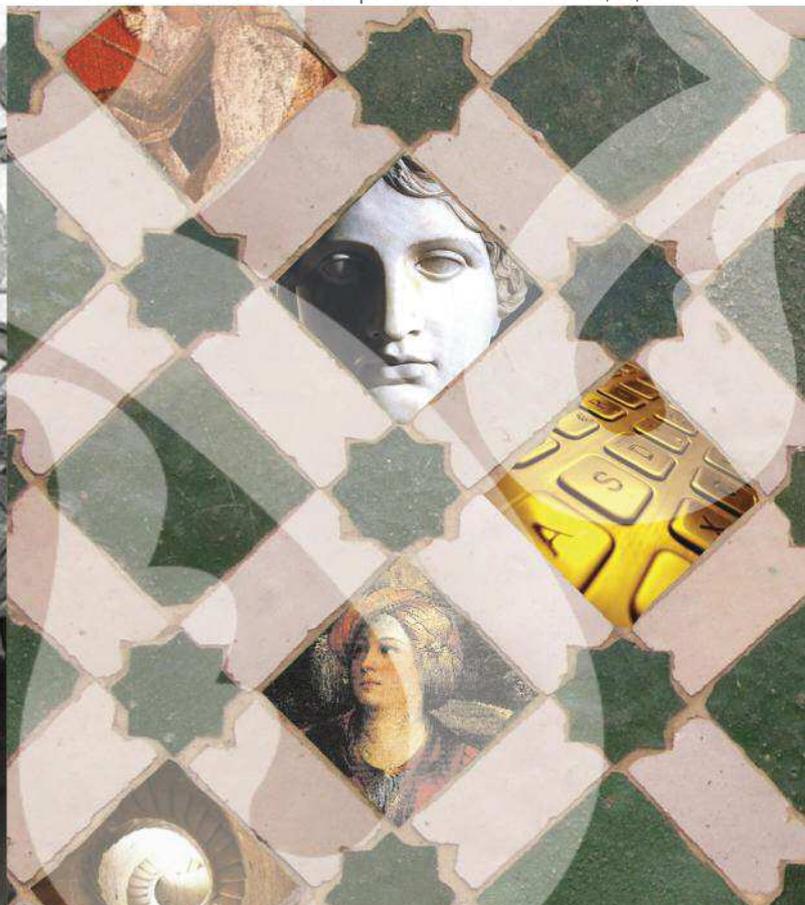
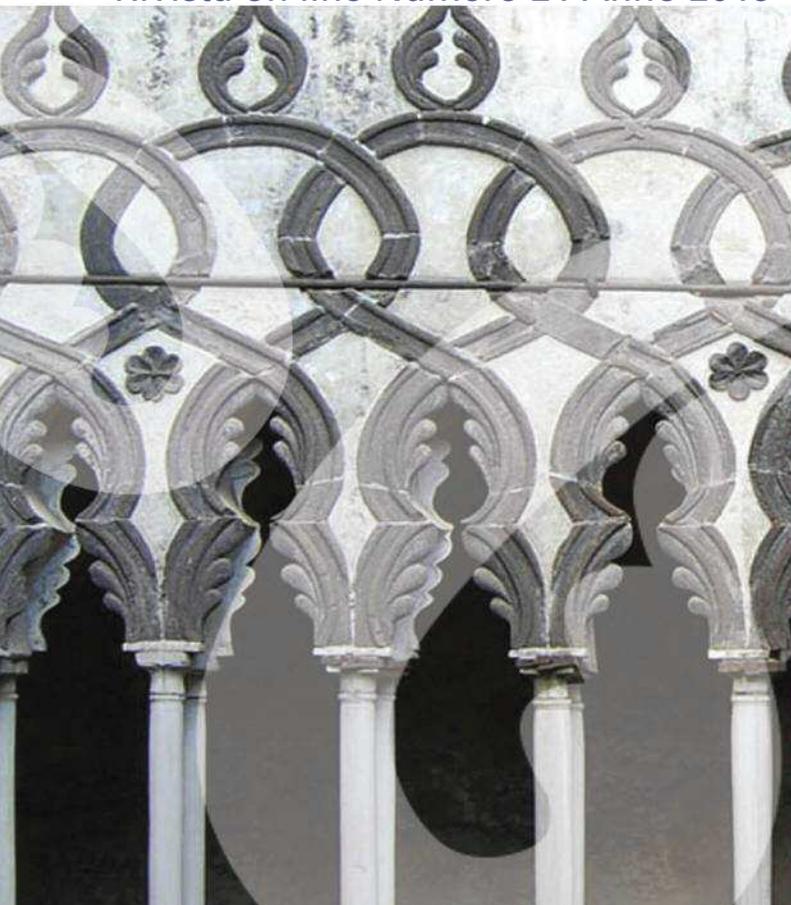


Centro Universitario Europeo  
per i Beni Culturali  
Ravello

# Territori della Cultura

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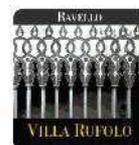
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# The Fibonacci Sequence and the Golden Section in a Lunette. Decoration of the Medieval Church of San Nicola in Pisa

## Abstract

The lunette under the arch of the original portal of the church of San Nicola in Pisa, Italy, dating from the 12th century, has an unusual linear geometric decoration that contains a tarsia, rarely mentioned in the literature and never interpreted. An original interpretation of its pattern based on a geometric reasoning is presented to show that the lunette represents the *golden section* with the linear geometric decoration and the *Fibonacci sequence* with the tarsia pattern.

## 1. The Church of San Nicola



Fig. 1 The Pellegrini brothers' model of the church.

The Benedictine church of San Nicola in Pisa, built in the mid-12th century and later enlarged and transformed by the Augustinians between 1297 and 1313, is famous for its particular octagonal bell tower, the most characteristic in Pisa after the Leaning Tower. It also houses various 13th century works, such as the painting of the Madonna and Child, by the Pisan Francesco Traini, the polychrome wooden crucifix attributed to Giovanni Pisano, an Annunciation by Nino Pisano, and the painting of St. Nicholas of Tolentino saving Pisa from the plague of the 15th century, with one of the first depiction of the city of Pisa (Fig. 1).

The façade has pilasters, blind arches and lozenges, and is decorated with tarsias of the 12th century. There is a lunette under the arch of the original portal of the church with an unusual linear geometric decoration that

contains a tarsia with squares and circles, made of green, white and red marble. This lunette has rarely been mentioned in the literature, and never interpreted (Fig. 2).

In February 2015, the peculiarities of the lunette attracted the attention of Pietro Armienti of the Department of Earth Sciences at the University of Pisa, who noted that the design contents seemed to have a geometric representation of the numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 of the Fibonacci sequence, in which each number other than 1 is the sum of the two previous numbers.



In this paper we use geometry to show that the lunette represents two themes:

(a) the calculation of the *golden section* with the linear geometric decoration around the tarsia and (b) the *Fibonacci sequence* with the design of the tarsia, thus confirming what Armienti had noticed.<sup>1</sup>

Firstly, we show how to build an *ideal model* of the tarsia pattern, and then how it can be refined in order to obtain the *final model* with the actual tarsia pattern. Finally, we interpret the particular linear geometric decoration of the lunette.

With no historical documentation on the subject, we do not know if it was intended that the two different themes were supposed to be read separately, because they represent two different concepts, or if in putting them together into a single lunette there was an implied relationship between them. We also do not know why it was decided to put the two themes under the arch of the original portal of the church.

## 2. The Tarsia Ideal Model

The *tarsia ideal model* is drawn with just a ruler and a compass, and *fine lines*, as follows, representing the Fibonacci sequence numbers  $n_i$  as circumferences  $C_{n_i}$  with the diameter proportional to the numbers of the sequence.

1. Let  $Q$  be the *colored tarsia square* circumscribed the circumference  $C_{55}$  (Fig. 3).



Fig. 2 The lunette with the tarsia.

<sup>1</sup> Armienti, P. (in press). The medieval roots of modern scientific thought. A Fibonacci abacus on the façade of the church of San Nicola in Pisa. *Journal of Cultural Heritage*. The author presents another interpretation of the tarsia as an abacus to draw sequences of regular polygons inscribed in a circle.



Fig. 3 The tarsia square and the circumference  $C_{55}$ .

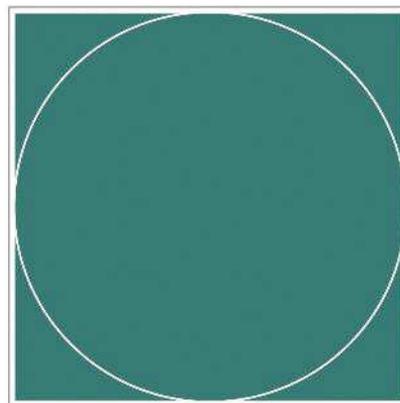
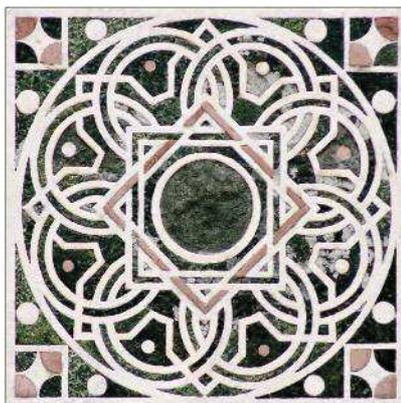


Fig. 4 The circumferences  $C_{55}$ ,  $C_{34}$ ,  $C_{21}$ ,  $C_{13}$  and the internal squares.

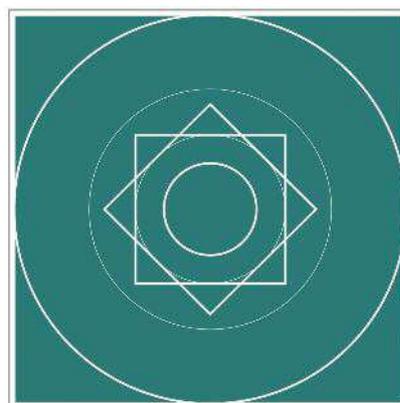
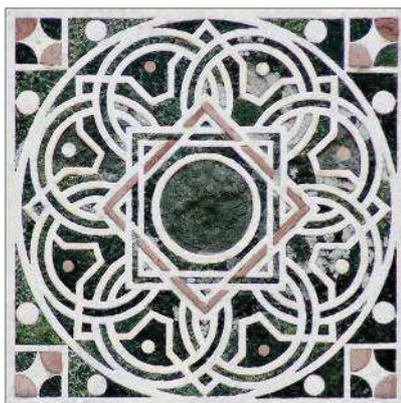
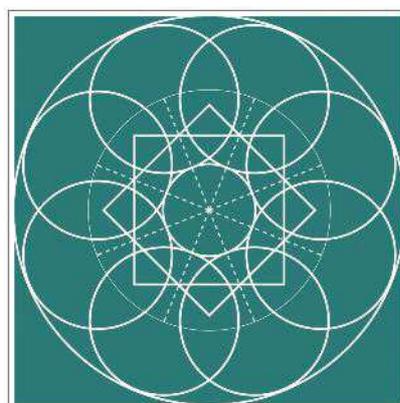
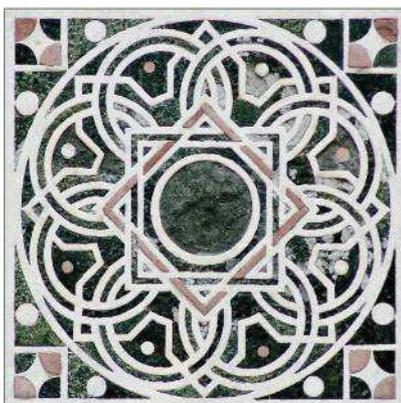


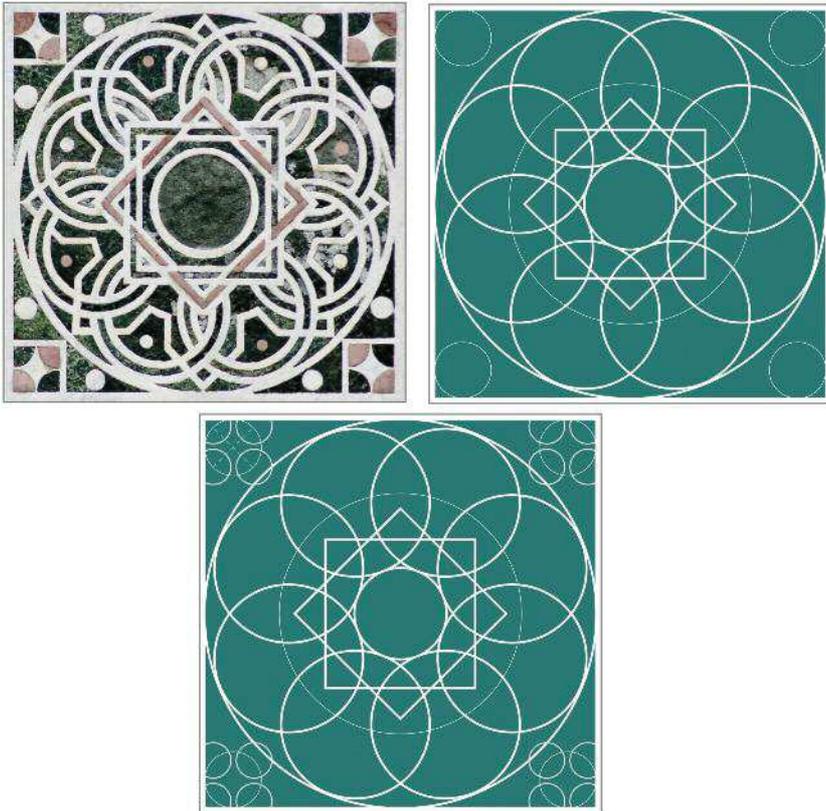
Fig. 5 The circumferences  $C_{21}$  internally tangent to  $C_{55}$ .



2. The circumferences  $C_{34}$ ,  $C_{21}$  and  $C_{13}$  are drawn concentric to  $C_{55}$ . The circumference  $C_{21}$  is used to draw a square  $Q_1$  and  $Q_2$  circumscribed, one rotated by 45 degrees from the other. The circumference  $C_{21}$  does not appear in the final ideal model of the tarsia. The intersection of the  $Q_1$  and  $Q_2$  produces an octagon  $O_1$  (Fig. 4). In the final ideal model of the tarsia, only some arcs of the circumference  $C_{34}$  appear.
3. Another eight circumferences  $C_{21}$  are drawn internally tangent to  $C_{55}$  with centers on the intersection of the diagonals of the octagon  $O_1$  with  $C_{34}$  (Fig. 5).
4. Four circumferences  $C_8$  are drawn inscribed in the corners of  $Q$ , and for each of them, four smaller circumferences  $C_5$



Fig. 6 The circumferences  $C_8$  and  $C_5$ .



are drawn centered on the intersections of the two 45 degrees diagonals with the circumference (Fig. 6).

5. Eight circumferences  $C_3$  are drawn internally tangential to the edges of the horizontal and vertical *golden rectangles* of the square are circumscribed to the circumference  $C_{34}$ . These two *golden rectangles* are inscribed in  $Q$ . The centers of the  $C_3$  are on the golden rectangles diagonals (Fig. 7).
6. Four circumferences  $C_2$  and  $C_1$  are drawn alternately with centers on the intersection of the diagonals of the octagon  $O_1$  with circumference  $C_{1-2}$  with diameter the average value of the diameters of  $C_{55}$  and  $C_{34}$ , i.e. a diameter half that of the circumference  $C_{89} = C_{55} + C_{34}$  (Fig. 8).

Fig. 7 The circumferences  $C_3$ .

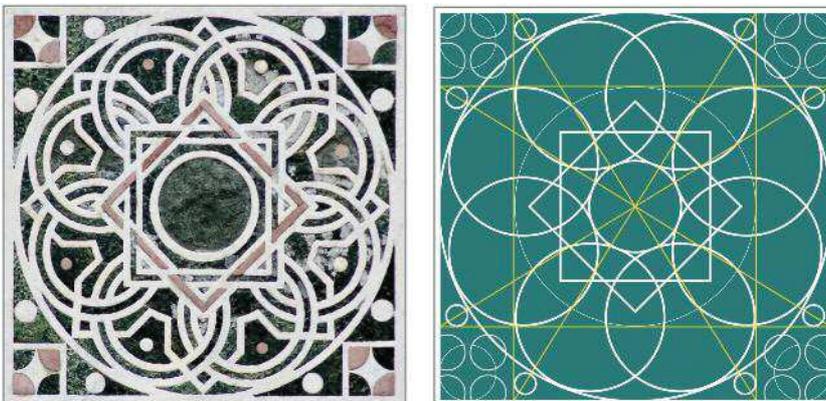




Fig. 8 The four circumferences  $C_1$  and  $C_2$ .

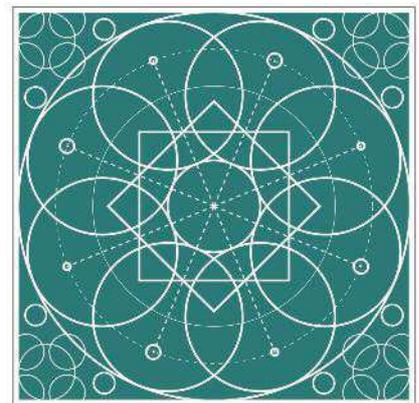
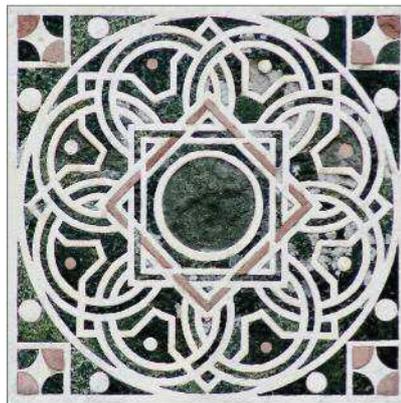


Fig. 9 The circumferences and squares based on the Fibonacci sequence.

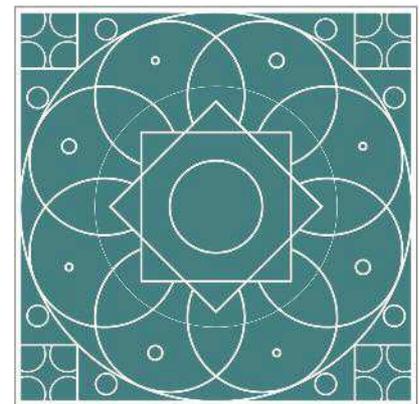
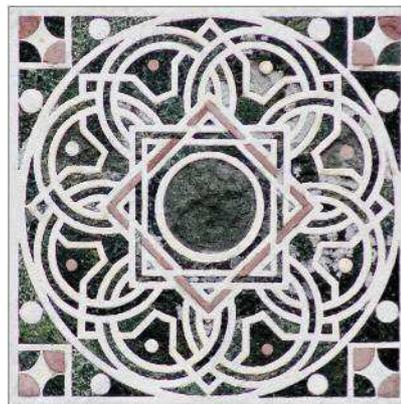
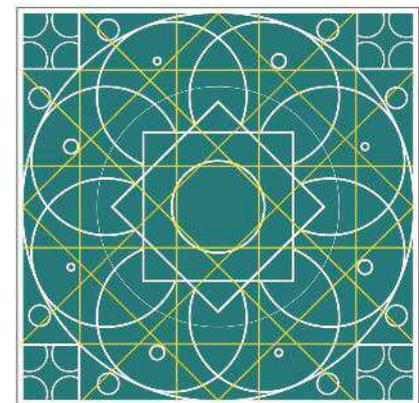
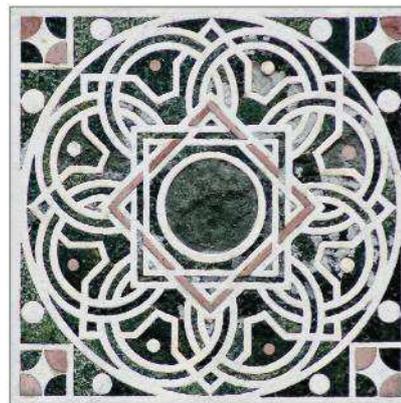


Fig. 10 The parallel lines to draw the pilaster decorations.



7. Four squares are drawn circumscribed to the circumferences  $C_8$ , which will not appear in the final ideal model of the tarsia (Fig. 9).
8. To draw the angled pilasters decoration of the tarsia, with eight pilasters as in the church bell tower, let us draw first (a) two squares inscribed in the circumference  $C_{55}$ , one rotated by 45 degrees from the other, (b) two pairs of diagonal parallel lines with ends on the vertices of the squares in the corners of the tarsia square, and (c) two pairs of horizontal and vertical parallel lines on the central axes at the same distance as the diagonal parallel lines (Fig. 10).
9. The angled pilaster lines are drawn from the vertices of the squares inscribed in the circumference  $C_{55}$  up to the parallel



Fig. 11 Circumferences, squares and angled pilasters of the ideal model of the tarsia.

lines drawn above. Then the lines continue on those parallel up to the circumference of  $C_{34}$ . Finally, the “close” lateral lines of the angled pilasters are connected with arches of the circumference  $C_{34}$ .

Once the parallel lines to draw the pilasters are eliminated, we get a preliminary version of the design of the ideal model of the tarsia (Fig. 11).

10. In order to draw a more precise and colored tarsia model, we proceed as follows (Fig. 12):

- a) For each circumference  $C_{21}$  tangent internally to  $C_{55}$ , an inner concentric circumference  $C_{17}$  is drawn of diameter the mean value of  $C_{21}$  and  $C_{13}$  diameters, i.e. half the diameter of the circumference  $C_{34} = C_{21} + C_{13}$ . Let  $\delta$  be the difference between the diameters of  $C_{21}$  and  $C_{17}$ .



Fig. 12 The final design of the ideal colored model of the tarsia.



Fig. 13 The colored ideal model with only the thickness of the lines changed.



b) In the two inner squares  $O_1$  and  $O_2$ , circumscribed to the circumference  $C_{21}$  with diameter  $\rho$ , two smaller squares are drawn of dimension  $(\rho - \delta)$ . In the figure the rotated square median line is drawn colored red as in the tarsia.

### 3. The Final Model of the Tarsia

Once the ideal model of the tarsia has been found, we then need to refine it to produce the final version by changing the *thickness* of the lines like the one used in the lunette. Let us assume that the thickness is equal to the diameter of the circumference  $C_1$ .

Drawing the ideal model of the colored tarsia just with thicker lines would produce an incorrect final model with some lines overlapping the circumference  $C_{55}$  (Fig. 13).

Instead, to draw a correct final model of the tarsia by having its pattern with thicker lines without any unwanted overlapping, we follow the steps below, assuming that these steps were probably at the basis of how the tarsia was originally carved out (Fig. 14):

1. The circumference  $C_{55}$  is drawn with the thickness on the inside. The colored tarsia square is circumscribed to the inner edge of  $C_{55}$ .
2. The angled pilaster decorations are drawn with the heads

Fig. 14 The tarsia and the final model.

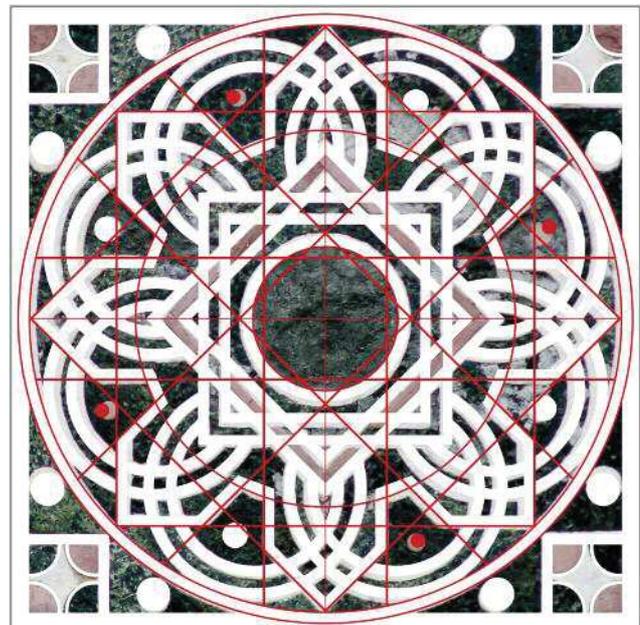




- angled moved towards the center, in order to have them *in-scribed* in the inner edge of the circumference  $C_{55}$ , and with thickness internal with respect to the pairs of the diagonal, horizontal and vertical parallel lines. The arcs of the angled pilaster decorations are drawn with the thickness on the inside of the circumference  $C_{34}$ .
3. The four squares in the vertices of the external square are drawn with a smaller size so that they will be in contact with the outer edge of the circumference  $C_{55}$ .
  4. The internal squares circumscribed to  $C_{21}$  and  $C_{17}$  are drawn with the thickness on the outside. The median line of the rotated square is colored red as in the tarsia.
  5. To draw the eight circumferences  $C_{21}$  internally tangent to  $C_{55}$  they are drawn with a smaller diameter using the circumference  $C_{20}$ , with the thickness on the inside and with the centers moved on  $C_{33}$  that passes through the middle of the arcs of the angled pilaster decorations. The eight circumferences  $C_{17}$  are drawn with a smaller diameter using the circumference  $C_{16}$  with the same centers of  $C_{20}$  and with the thickness on the inside.
  6. The circumference  $C_{13}$  is drawn with the thickness on the outside.
  7. The circumferences  $C_1$  and  $C_2$  are drawn internally tangent to  $C_{1-2}$ .
  8. The intersections of the circumferences  $C_{20}$  and  $C_{16}$  with the angled pilasters, as well as those of the internal squares sides, are drawn with interweaving patterns as in the tarsia of the lunette to give the illusion of lines weaving over and under each other.

Figure 15 shows the final model of the tarsia superimposed on the original carved pattern. There is a some lack of fit due to possible inaccuracies both in the construction of the original tarsia (for example, the external circumference is not perfect) and in the way it has recently been restored without knowledge of its meaning (for example, the circumferences  $C_1$  have different diameters). However, the final model of the tarsia pattern and the original one carved are close enough to validate the correctness of the model for its proposed interpretation.

Fig. 15 The final model of the tarsia superimposed on the original carved pattern.





#### 4. The Linear Geometric Decoration of the Lunette

In this section we show that the unusual linear geometric decoration of the lunette is based on the *golden section*, which appears already in Euclid's *Elements*, where he does not use this term but speaks of *division in extreme and mean ratio*. In addition, the central rectangle that contains the tarsia is the *golden rectangle* of the tarsia square and the rest of the lunette around the *golden rectangle* is decorated with lines parallel to the diagonals of the tarsia, by producing triangles, squares and rectangles.

In the following, we briefly recall the geometric construction of the golden section of a segment and of the golden rectangle of a square. We then show how to build an ideal model of the linear geometric decoration of the lunette.

##### **The Golden Section of a Segment**

In his paper "Some Episodes in the Life and Times of Division in Extreme and Mean Ratio, in E. Giusti, *Luca Pacioli e la Matematica del Rinascimento*, Petrucci Editore, Città di Castello, 1998" David Fowler, an expert in the history of mathematics in ancient Greece, recalls that one way of building the golden section of a segment  $AB$  with a ruler and a compass was described in Euclid's *Elements* (c. 300 BC) and a variant of his solution, later commonly used, was conceived by Hero of Alexandria (c. 10 - c. 70 BC), as al-Nayrizi recalls in a Latin translation of scientific works of the Italian Gerard of Cremona (Cremona, 1114 - Toledo, 1187) (Fig. 16).

First the design of a right triangle is drawn with a side  $AB$  and the other side  $BC$  of length  $AB/2$ . Then the arc  $BS$  of the circumference with center  $C$  and radius  $BC$  is drawn and the length of the segment  $AS = AH$  is the *golden section* of  $AB$ : the point  $H$  cuts  $AB$  in such a way that  $AB$  stands to the bigger part  $AH$  in the same proportion as the bigger part to the smaller part  $HB$ .

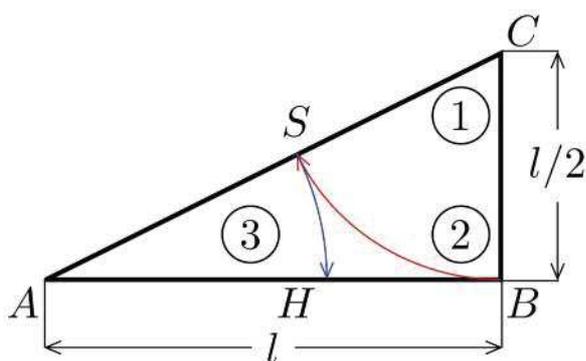


Fig. 16 The golden section of the segment  $AB$  with a ruler and a compass.

##### **The Golden Rectangle of a Square**

Another problem presented by Euclid is the geometric construction of a *golden rectangle* of a square  $ABCD$  with the



method shown in Figure 17. A golden rectangle is one in which the smaller side is the golden section of the bigger side.

First  $ABCD$  is drawn with the length of the side equal to the smaller side of the rectangle. Let  $E$  be the midpoint of  $AB$  and  $EC$  be the radius of the circumference with center  $E$  crossing  $AB$  extended at  $G$ :  $AGFD$  is the golden rectangle of  $ABCD$  and the length of the segment  $BH = BG$  is the golden section of  $BC$ . Another interesting property is that the rectangles  $BGFC$  and  $ABHI$  are equal and are golden rectangles.

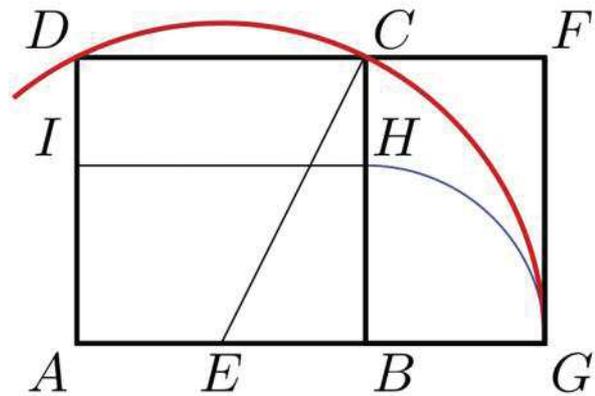


Fig. 17 The construction of the golden rectangle of the square  $ABCD$ .

### The Model of the Lunette Decoration

The shape of the lunette on the original portal architrave, shown in Figure 18, is a stilted arch with dimensions determined when the original portal of the church was built. The linear geometric decoration of the lunette is based on the golden rectangle  $R$  of the inner tarsia square, designed on the center of the lunette base. We can imagine that the artist of the geometric decoration had proceeded as follows.

#### 1. Dimensions of the golden rectangle $R$ (Fig. 18).

- a) The semicircle of the lunette is drawn with the center  $M$  and radius  $MC$ .
- b) The square  $ABIN$  is drawn inscribed in the semicircle geometrically. Among the possible solutions, we show the following one based on the method described in Figure 17: (a) the triangle  $MCH$  is drawn on the center of the semicircle with the side  $HM$  equal to half of  $MC$ ; (b) from the center  $M$  the segment  $MI$  perpendicular to the hypotenuse is drawn that intersects the semicircle in the point  $I$ , which is a vertex of the square  $ABIN$ .
- c)  $GB$  is the golden section of the side  $IB$  of  $ABIN$ , as it has been shown in Figure 17, and  $ABGL$  is the golden rectangle  $R$ .

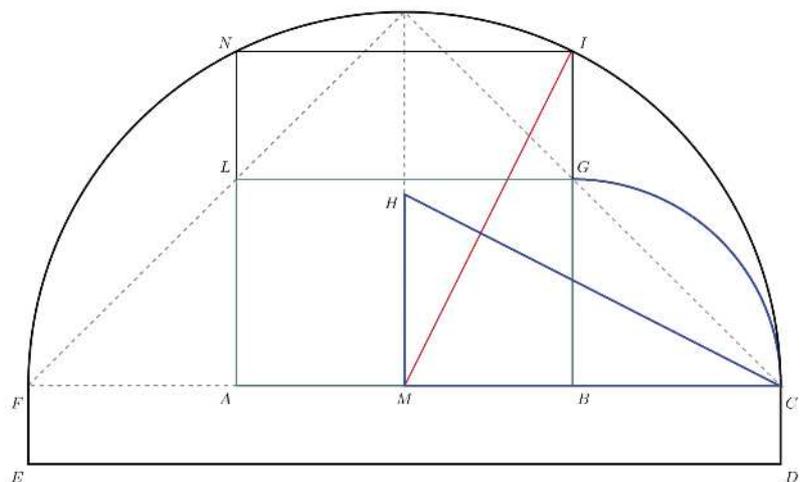


Fig. 18 The design of a golden rectangle inside the lunette semicircle.



2. *Design of the golden rectangle  $R$  on the lunette base (Fig. 19)*

- a) The golden rectangle  $ABGL$  of Figure 18 is drawn in contact with the base of the lunette.
- b) The diagonals are then used to draw the triangles and squares of the linear geometric decoration.

3. *Design of the linear geometric decoration of the lunette around the golden rectangle  $R$  (Fig. 20).*

- a) The internal tarsia square, whose side is equal to the shortest side of  $R$ , is drawn inscribed within golden rectangle.
- b) The lines parallel to the diagonals of the tarsia produce in the part of the lunette above  $R$  two isosceles triangles with 45 degree angles, each half of a square  $q$  with a diagonal  $d$  of length equal to half the length of  $R$  and height  $d/2$ , and squares with the same sizes of  $q$ .
- c) Two decorations appear (a) a red mark in the middle of the side of a right triangle and (b) two  $M$ , each obtained by combining two triangles with one side twice the length of the other. The red mark is also on the midpoint of a square side thus recalling Euclid's method of building both the golden section of a segment and the construction of the golden rectangle of a square.
- d) On the shorter sides of the central golden rectangle  $R$  two triangles are also drawn, larger than those drawn over  $R$ . In the lunette model the sides of these two triangles are equal to the golden section of the shorter side of  $R$ . The lunette consequently has some squares, which then become rectangles. This may have been done in order to highlight a law of growth, as with the Fibonacci sequence used to draw the tarsia.

Fig. 19 The design of the golden rectangle on the lunette base.

Fig. 20 An ideal model of the linear geometric decoration of the lunette.

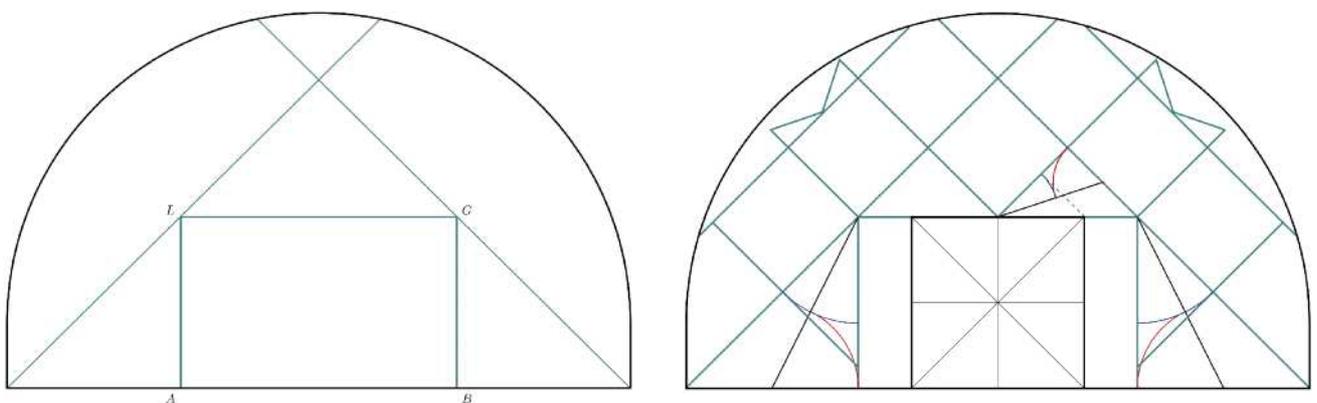




Figure 21 shows the final model of the lunette.

Finally, the lunette with the tarsia devoted to the sequence of the “Fibonacci numbers”, described in his book *Liber Abaci* (1202), recalls the solution of the “rabbit problem”, a “recreational” mathematical problem regarding the rate of growth of a hypothetical colony of rabbits based on idealized assumptions.

The lunette thus recalls both the concepts of *beauty* and *harmony*, with the geometric construction of the golden section of a segment, and the *procreation*. Eight actions of procreation are modeled with the nine Fibonacci numbers represented, but eight is also the symbol for infinity.

## 5. Conclusions

We have shown a geometric model of the lunette under the arch of the original portal of the church of San Nicola in Pisa. The model highlights that the lunette represents the *golden section* with a linear geometric decoration and the *Fibonacci sequence* with a tarsia pattern. It has thus been revealed that in Pisa there is a church that recalls Fibonacci (a native to Pisa) with a tribute of a tarsia in the original portal lunette.

When the lunette was originally created, the relationship between the two themes was not known because Fibonacci did not study the mathematical properties of the sequence, which had actually been used in the 6th century by the Indian mathematician Virahanka’s analysis of the poetic meter of Indian poems. It was not until 1611 that J. Kepler showed that the ratio of two successive numbers in the sequence is alternately higher and lower than the golden section and that, with the progress of sequence, the golden section is the limit of this ratio. Therefore, without the designers of the lunette even being aware of it, they created a fascinating relationship between the tarsia devoted to the Fibonacci sequence and the linear geometric decoration devoted to the golden section.

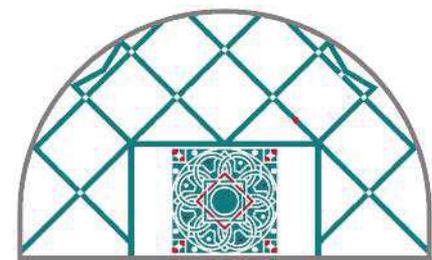
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(a) The lunette



(b) The model

Fig. 21 The lunette devoted to the golden section and to the Fibonacci sequence.